11. THE ALEXANDER MODULE

§11.1. Modules over the Ring of Rational Laurent Polynomials

There's an even better invariant for a link than the Alexander Group – the Alexander Module. But before we can discuss it we need to learn some more algebra. We all know about polynomials. A polynomial, in λ, is a formal expression of the form:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0.$$

where the a_i's are the coefficients and λ is an 'indeterminate'.

(I'll use λ rather than the traditional *x*.)

The coefficients can come from any field, but we'll restrict ourselves to the case where they are rational numbers. These rational polynomials form a ring under

the usual operations of addition and multiplication of polynomials. A **ring** is a mathematical structure with two binary operations + and × satisfying a whole bunch of axioms. You should know the field axioms from having done a linear algebra course. The one field axiom that breaks down for rational polynomials is the axiom that says that every non-zero element has an inverse under multiplication. The only rational polynomials that *do* have multiplicative inverses in $\mathbb{Q}[\lambda]$ are the non-zero constant polynomials ±1.

Now there's no good reason why we can't include negative powers of λ. Of course expressions such as

$$\lambda^2 + 2\lambda + \lambda^{-3}$$

aren't polynomials, but we can still add and multiply them as we do ordinary polynomials. We call them Laurent polynomials.

A **Laurent polynomial**, over $\mathbb{Q}$, is a formal expression of the form:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_m\lambda^m$$

where the a_i's are integers and where m, n are integers, possibly negative, with $n \ge m$. We'll only be considering Laurent polynomials over $\mathbb{Q}$ so from now on we drop the qualifier 'over $\mathbb{Q}$' and assume that this is always the case.

Example 1: $2\lambda + 3 + \lambda^{-2}$ and $\lambda^3 + 4\lambda - 5\lambda^{-1}$ are Laurent polynomials.

Their sum is $(2\lambda + 3 + \lambda^{-2}) + (\lambda^3 + 4\lambda - 5\lambda^{-1})$

$$= \lambda^3 + 6\lambda + 3 - 5\lambda^{-1} + \lambda^{-2}$$

and their product is $(2\lambda + 3 + \lambda^{-2})\,(\lambda^3 + 4\lambda - 5\lambda^{-1})$

$= (2\lambda^4 + 8\lambda^2 - 10) + (3\lambda^3 + 12\lambda - 15\lambda^{-1})$

$+ (\lambda + 4\lambda^{-1} - 5\lambda^{-3})$.

$= 2\lambda^4 + + 3\lambda^3 + 8\lambda^2 + 13\lambda - 10 \quad - 11\lambda^{-1} - 5\lambda^{-3}$

The set of all rational Laurent polynomials is denoted by $\mathbb{Q}(\lambda)$. It contains the set of all rational polynomials, namely $\mathbb{Q}[\lambda]$. (Square brackets mean polynomials, with no negative powers. Round brackets include negative powers of λ.)

Even though we are now including negative powers of λ, $\mathbb{Q}(\lambda)$ is still not a field. In fact the only Laurent polynomials with inverses in $\mathbb{Q}(\lambda)$ are the powers of λ.

Now an abelian group is rather similar to a vector space. Remember that in a vector space you can add any two vectors and the axioms for addition are precisely those for an abelian group. But in a vector space there's the operation of multiplication by a scalar, with several axioms regulating this operation. These additional axioms, for a vector space V over the field F, are:

- $a\mathbf{v} \in V$ for all $\mathbf{v} \in V$ and $a \in F$;
- $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ for all $\mathbf{v} \in V$ and $a, b \in F$;
- $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $a \in F$.
- $1\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

Abelian groups don't have scalars – at least there was no mention of them in the definition. This is true, but remember that we defined *ng* for any integer *n* and any group element *g*. We can therefore consider the integers as being scalars. But $\mathbb{Z}$ isn't a field. That's also true and that's why abelian groups aren't vector spaces. But notice that all four of the above vector space axioms hold in an abelian group if the scalars come from $\mathbb{Z}$. Let's write them out again, this time using notation that's more appropriate for an abelian group G.

- ng $\in$ G for all g $\in$ G and $n \in \mathbb{Z}$;
- $(m + n)$g = mg + ng for all g $\in$ G and $m, n \in \mathbb{Z}$;
- n(g + h) = ng + nh for all g, h $\in$ G and $n \in \mathbb{Z}$;
- 1g = g for all g $\in$ G.

What we're going to do is to extend our ring of scalars to include not just integers, but all rational Laurent polynomials. And, as with our abelian groups, we're going to consider formal linear combinations.

So a typical formal linear combination will have the form: $a_1(\lambda)\boldsymbol{x}_1 + \ldots + a_n(\lambda)\boldsymbol{x}_n$ where the coefficients, $a_i(\lambda)$ are Laurent polynomials and where the $\boldsymbol{x}_i$ are indeterminates.

We'll denote the set of all such formal linear combinations of the indeterminates $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, over $\mathbb{Q}(\lambda)$, by $\mathbb{Q}(\lambda)[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$. We can add such expressions together in the obvious way, and even multiply them by a scalar from $\mathbb{Q}(\lambda)$. Moreover all the vector space axioms hold, apart from the small detail that the ring of scalars, $\mathbb{Q}(\lambda)$, isn't a field. We call such structures **$\mathbb{Q}(\lambda)$-modules**.

Example 2: Suppose that $\boldsymbol{a} = (3 + \lambda^{-2})\,\boldsymbol{x} + (4\lambda^3 + 5\lambda^{-1})\,\boldsymbol{y}$ and $\boldsymbol{b} = (7 + 3\lambda^{-4})\,\mathbf{x} + (5\lambda + 3\lambda^{-1})\,\boldsymbol{y}$.
Then $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Q}(\lambda)[\boldsymbol{x}, \boldsymbol{y}]$.

Addition:
$\boldsymbol{a} + \boldsymbol{b} = (10 + \lambda^{-2} + 3\lambda^{-4})\, \boldsymbol{x} + (4\lambda^3 + 5\lambda + 8\lambda^{-1})\, \boldsymbol{y}$
Scalar Multiplication:
$(\lambda - \lambda^{-1})\mathbf{a} = (\lambda - \lambda^{-1})(3 + \lambda^{-2})\boldsymbol{x} + (\lambda - \lambda^{-1})(4\lambda^3 + 5\lambda^{-1})\boldsymbol{y}$
$= (3\lambda - 2\lambda^{-1} - \lambda^{-3})\boldsymbol{x} + (4\lambda^4 + 5 - 4\lambda^2 - 5\lambda^{-2})\boldsymbol{y}$

Multiplication: ***ab*** is not defined since ***xy*** has no meaning. Remember that $\mathbb{Q}(\lambda)[\boldsymbol{x}, \boldsymbol{y}]$ is a module, not a ring. In fact $\mathbb{Q}(\lambda)[\boldsymbol{x}, \boldsymbol{y}]$ is much more like an abelian group than a ring.

Just as we have abelian groups defined by means of generators and relations so we have $\mathbb{Q}(\lambda)$-modules defined in this way.

We use the notation $[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \mid R_1, \ldots, R_m]$, where each R_i is a linear combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ with coefficients coming from $\mathbb{Q}(\lambda)$, to denote the corresponding $\mathbb{Q}(\lambda)$ module. The coefficients in the relations R_i are Laurent polynomials, but clearly we can rewrite them as polynomials.

For example we can replace

$(2\lambda + \lambda^{-1})\boldsymbol{x} + (-7\lambda^5 + 3\lambda^{-2})\boldsymbol{y} = 0$ by multiplying by λ^2 to give $(2\lambda^3 + \lambda)\boldsymbol{x} + (-7t^7 + 3)\boldsymbol{y} = 0$.

Example 3:
$[\boldsymbol{x}, \boldsymbol{y} \mid (\lambda^2 + 1)\boldsymbol{x} + 2\lambda^3\boldsymbol{y} = 0, (\lambda + 2\lambda^3)\, \boldsymbol{x} + (\lambda + 3)\boldsymbol{y} = 0]$ is a $\mathbb{Q}(\lambda)$-module.

Now the theory of modules over a ring is similar to that of vector spaces (modules over a field) or abelian groups (modules over $\mathbb{Z}$). But there are some important differences.

- Finitely generated vector spaces are direct sums of cyclic subspaces – in other words they have bases and their number (the dimension) is always the same for a given vector space. Another way of saying this is that every finite-dimensional vector space is isomorphic to a direct sum of cyclic subspaces (dimension 1) and the number of direct summands for a given vector space is constant.

- Finitely generated abelian groups are direct sums of cyclic subgroups but the number of summands is not constant. For example $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$. So the concept of dimension doesn't apply here.

- Finitely generated modules over a ring, even the ring $\mathbb{Q}(\lambda)$, need not even be a direct sum of cyclic submodules.

	Finitely generated		
	Vector spaces	**Abelian groups**	**Modules**
direct sum of 1-dimensionals	YES	YES	NO
have a unique dimension	YES	NO	NO

Example 4: $[\boldsymbol{x} \mid (\lambda^2 + 2\lambda + 3)\boldsymbol{x} = 0]$ is a cyclic $\mathbb{Q}(\lambda)$-module. It is therefore an abelian group, but as such, it is not cyclic.
Since $\lambda^2\boldsymbol{x} = -2\lambda\boldsymbol{x} - 3\boldsymbol{x}$ we may conclude that
$$\lambda^3\boldsymbol{x} = -2\lambda^2\boldsymbol{x} - 3\lambda\boldsymbol{x} = -2(-2\lambda\boldsymbol{x} - 3\boldsymbol{x}) - 2\lambda\boldsymbol{x}$$
$$= -3\boldsymbol{x} + 2\lambda\boldsymbol{x}.$$
In the same way we can express all positive powers in terms of $\boldsymbol{x}$ and $\lambda\boldsymbol{x}$.

Now $3\boldsymbol{x} + 2\lambda\boldsymbol{x} + \lambda^2\boldsymbol{x} = 0$ so:
$$\boldsymbol{x} = -\tfrac{1}{3}\lambda^2\boldsymbol{x} - \tfrac{2}{3}\lambda\boldsymbol{x} \text{ whence}$$
$$\lambda^{-1}\boldsymbol{x} = -\tfrac{1}{3}\lambda\boldsymbol{x} - \tfrac{2}{3}\boldsymbol{x} \text{ and}$$
$$\lambda^{-2}\boldsymbol{x} = -\tfrac{1}{3}\boldsymbol{x} - \tfrac{2}{3}\lambda\boldsymbol{x}$$
Continuing we can express all negative powers in terms of $\boldsymbol{x}$ and $\lambda\boldsymbol{x}$. So as a vector space over $\mathbb{Q}$ this $\mathbb{Q}(\lambda)$ module has dimension 2, with $\{\boldsymbol{x}, \lambda\boldsymbol{x}\}$ as a basis.

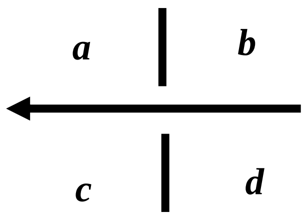
§11.2. The Alexander Module of an Oriented Knot

An **oriented knot** is a knot with one of the two possible orientations specified. We can mark the orientation by arrows.

Suppose that K is an oriented knot. Take a projection of the link and assign a generator to each face. (So far we are proceeding as for the Face Group.) These are the generators for our module.

We obtain a relation at each of the crossings as follows. View the crossing so that the orientation of the overpass is from right to left. Then, if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are the generators corresponding to the faces surrounding the crossing, as indicated in the following diagram, we take the relation

$$\lambda\,(\boldsymbol{a} + \boldsymbol{b}) + (\boldsymbol{c} + \boldsymbol{d}) = \mathbf{0}.$$



Note the direction of the arrow. Always orient the picture (in reality or in your mind) so that the arrow goes from East to West. Then the λ goes with those faces to the

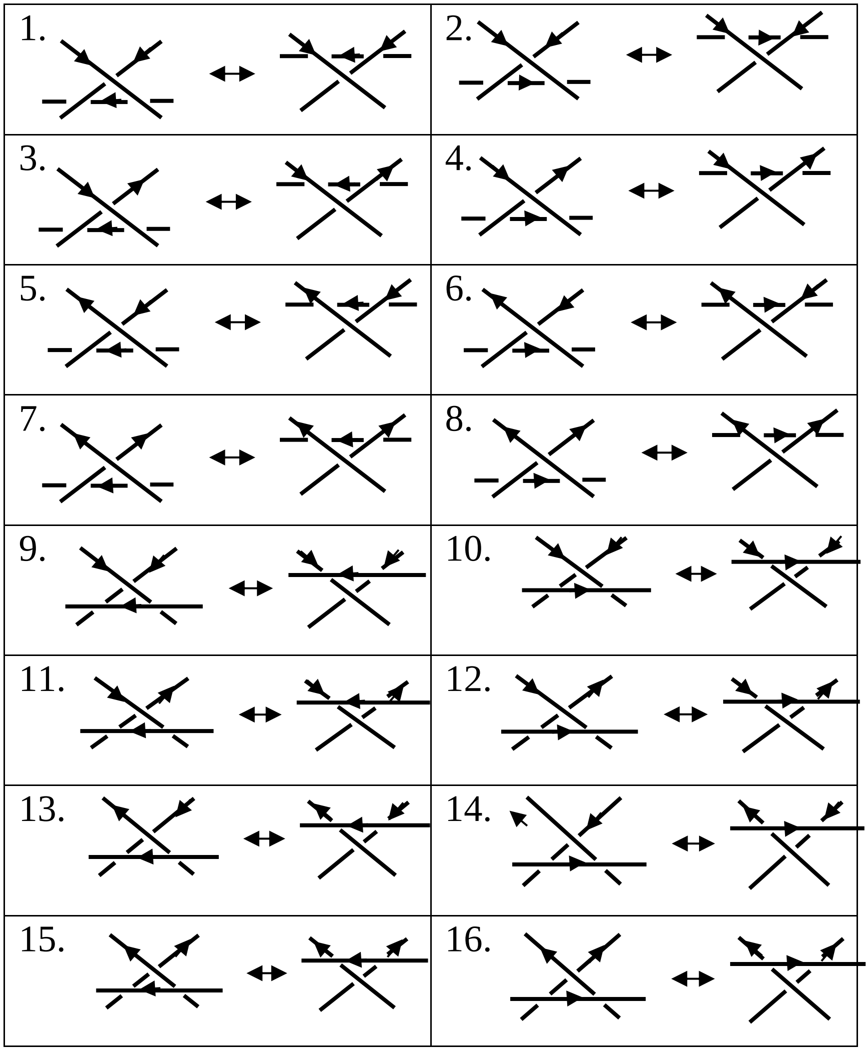
North of the unbroken line. One of the biggest sources of error is getting this wrong.

If $\lambda = -1$ this becomes the corresponding relation for the face group and the orientation doesn't matter. However, for the Alexander Module, we treat λ as an indeterminate. The resulting algebraic structure is a module over $\mathbb{Q}(\lambda)$, the ring of integer Laurent polynomials and we call it the **Face Module** of the oriented knot.

As we did with face groups we put the generators for two adjacent faces equal to zero and introduce generators only as required. So long as we have labels for three out of four faces around a crossing we can express the generator in terms of the 4^{th} face in terms of them. Otherwise we introduce a new generator. We call the module the **Alexander Module** of the oriented knot.

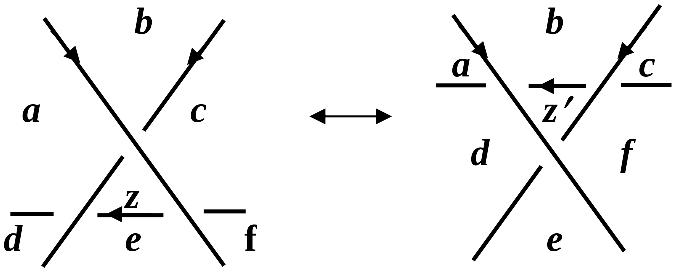
Theorem 1: The Alexander module is an invariant of an oriented knot.

Proof: To show this we simply follow the procedure detailed in Theorem 1 of chapter 10: we show that the Alexander Module is unchanged by each of the three Reidemeister moves. However we have to be careful about orientations. With the Reidemeister type III move there are 8 possible orientations of the three strands plus, for each, the two possibilities that the strand that lies entirely one side of the main crossing is above or below the other two strands. This gives 16 cases in all.



But without loss of generality we can assume that the horizontal strand lies below the others (for we can turn the knot over if it is not) and we can assume that the top-left-bottom-right strand goes down (for we can rotate the knot through 180° if it goes up). This means that we only need to consider cases 1-4. For each of these we can carry out a similar analysis to that which we did for the Alexander Group to show that we get an equivalent set of equations when such a Reidemeister type III move is carried out.

This is very tedious so I will only do this for case 1 and leave the rest as an exercise.



The equations for the left are:

$$\left.\begin{aligned} \lambda(\boldsymbol{a} + \boldsymbol{z}) + \boldsymbol{b} + \boldsymbol{c} &= 0 \\ \lambda(\boldsymbol{a} + \boldsymbol{d}) + \boldsymbol{e} + \mathbf{z} &= 0 \\ \lambda(\boldsymbol{e} + \boldsymbol{z}) + \boldsymbol{c} + \boldsymbol{f} &= 0 \end{aligned}\right\}$$

Equivalently:

$$\left.\begin{aligned} \lambda\mathbf{z} &= -\lambda\mathbf{a} - \mathbf{b} - \mathbf{c} \\ \lambda(\mathbf{a} + \mathbf{d}) + \mathbf{e} + \mathbf{z} &= 0 \\ \lambda^2(\boldsymbol{a} + \boldsymbol{d}) &= \boldsymbol{c} + \boldsymbol{f} \end{aligned}\right\}$$

Equivalently:

$$\left.\begin{aligned} \boldsymbol{z} &= -\boldsymbol{a} - \lambda^{-1}\boldsymbol{b} - \lambda^{-1}\boldsymbol{c} \\ \lambda\boldsymbol{a} + \lambda\boldsymbol{d} + \boldsymbol{e} &= \boldsymbol{a} + \lambda^{-1}\boldsymbol{b} + \lambda^{-1}\boldsymbol{c} \\ \lambda^2(\boldsymbol{a} + \boldsymbol{d}) &= \boldsymbol{c} + \boldsymbol{f} \end{aligned}\right\}$$

The middle equation can be rewritten as:

$$\lambda^{-1}(\boldsymbol{c} + \boldsymbol{f}) + \boldsymbol{e} = \boldsymbol{a} + \lambda^{-1}\boldsymbol{b} + \lambda^{-1}\boldsymbol{c} \text{ or}$$
$$\lambda^{-1}\boldsymbol{f} + \boldsymbol{e} = \boldsymbol{a} + \lambda^{-1}\boldsymbol{b} \text{ or}$$
$$\boldsymbol{f} + \lambda\boldsymbol{e} = \lambda\boldsymbol{a} + \boldsymbol{b}.$$

For the diagram on the right we have the equations:

$$\left.\begin{aligned} \lambda(\boldsymbol{a} + \boldsymbol{d}) + \boldsymbol{b} + \boldsymbol{z}' &= 0 \\ \lambda(\boldsymbol{b} + \boldsymbol{z}') + \boldsymbol{c} + \boldsymbol{f} &= 0 \\ \lambda(\boldsymbol{d} + \boldsymbol{e}) + \boldsymbol{z}' + \boldsymbol{f} &= 0 \end{aligned}\right\}$$

Equivalently:

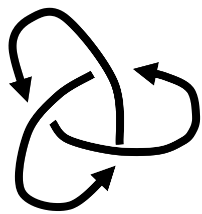
$$\left.\begin{aligned} \lambda(\boldsymbol{a} + \boldsymbol{d}) + \boldsymbol{b} + \boldsymbol{z}' &= 0 \\ \lambda^2(\boldsymbol{a} + \boldsymbol{d}) &= \boldsymbol{c} + \boldsymbol{f} \\ \boldsymbol{z}' &= -\lambda\boldsymbol{d} - \lambda\boldsymbol{e} - \boldsymbol{f} \end{aligned}\right\}$$

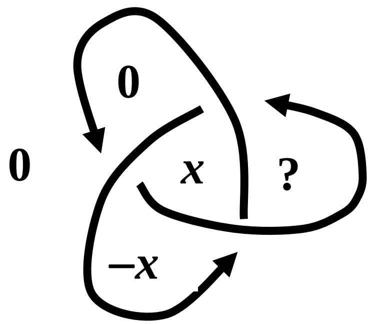
Equivalently:

$$\left.\begin{aligned} \lambda\boldsymbol{a} + \boldsymbol{b} &= \lambda\boldsymbol{e} + \boldsymbol{f} \\ \lambda^2(\boldsymbol{a} + \boldsymbol{d}) &= \boldsymbol{c} + \boldsymbol{f} \\ \boldsymbol{z}' &= -\lambda\boldsymbol{d} - \boldsymbol{e} - \boldsymbol{f} \end{aligned}\right\}$$

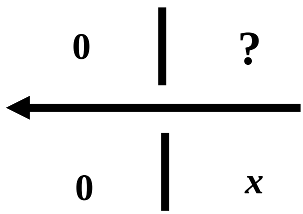
With the exception of the equations expressing $\boldsymbol{z}$ and $\boldsymbol{z}'$ these equations are identical. Now $\boldsymbol{z}$ and $\boldsymbol{z}'$ will not appear in any other equation arising from any other crossing and so we may remove $\boldsymbol{z}$ and $\boldsymbol{z}'$ as generators from the respective presentations, together with these equations that express them in terms of the other generators. So the face module does not change, up to isomorphism, as a result of this type III Reidemeister move. The other cases can be checked, as can the type I and type II moves.

Example 5: Find the Alexander Module of the following oriented trefoil knot.
Solution: Using the crossing adjacent to the two zeros we assign the labels ***x*** and –***x*** to the other two faces (it doesn't matter which way round we do it).





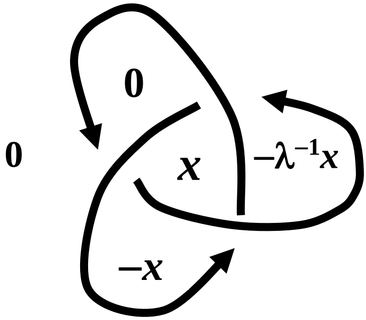
We view the topmost crossing as

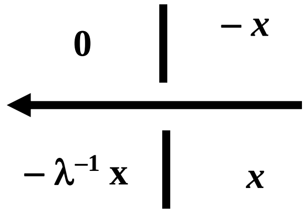


which gives $\lambda(\mathbf{0} + ?) + \mathbf{0} + \boldsymbol{x} = 0$ and hence $? = -\lambda^{-1}\boldsymbol{x}$.
At the remaining crossing we get the relation

$$\lambda(\mathbf{0} - \boldsymbol{x}) - \lambda^{-1}\boldsymbol{x} + \boldsymbol{x} = \mathbf{0}.$$

This can be simplified to $(\lambda - 1 + \lambda^{-1})\boldsymbol{x} = \mathbf{0}$.





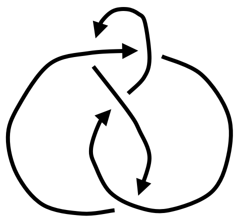
Since λ has an inverse in the ring of Laurent polynomials we are permitted to multiply or divide by powers of λ. Here we choose to multiply by λ, obtaining

$$(\lambda^2 - \lambda + 1)\boldsymbol{x} = \mathbf{0}.$$

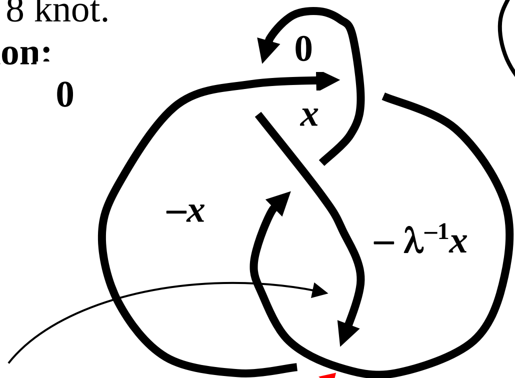
So the Alexander Module of this oriented trefoil knot is

$$[\boldsymbol{x} \mid (\lambda^2 - \lambda + 1)\boldsymbol{x} = \mathbf{0}] .$$

Example 6: Find the Alexander Module of the following oriented figure 8 knot.



Solution:



$(1 + \lambda^{-1} - \lambda^{-2})\boldsymbol{x}$ Here $(\lambda - 3 + \lambda^{-1})\boldsymbol{x} = 0$.

Hence $(\lambda^2 - 3\lambda + 1)\boldsymbol{x} = \mathbf{0}$.

The Alexander Module of this oriented figure 8 knot is thus $[\boldsymbol{x} \mid (\lambda^2 - 3\lambda + 1)\boldsymbol{x} = \mathbf{0}]$.

Theorem 2: The Alexander modules of a knot for each of the two orientations are isomorphic.
Proof: We omit the proof.

Hence we unambiguously define the Alexander Module for a knot to be the Alexander Module of either orientation.

§11.3. The Alexander Polynomial of a Knot

It can be shown that any two $\mathbb{Q}(\lambda)$ modules, given in terms of generators and relations, are isomorphic if and only if their relation matrices are similar via an invertible matrix over $\mathbb{Q}(\lambda)$. That is, $[A] \cong [B]$ if and only if $A = S^{-1}BS$ for some $\mathbb{Q}(\lambda)$ matrix, S, that has an inverse, S^{-1}, which is also a $\mathbb{Q}(\lambda)$ matrix.

So far we've been lucky to end up with just one generator and so our modules have been cyclic (with one generator). The relation matrix in such cases is 1×1 and the only possibilities for S are $\pm \lambda^r$ for some integer r. If we insist on multiplying by such a factor we can arrange for the sole entry in the 1×1 relation matrix to be a polynomial with positive leading coefficient and non-zero constant term. This is called the Alexander Polynomial.

But we need to define the Alexander Polynomial when the module is not cyclic, or when we can't make it so.

The **Alexander Polynomial** of a knot is the determinant of the relation matrix. Since similar matrices have the same determinant this is an invariant. It is normalised so that it is a polynomial with positive leading coefficient and non-zero constant term. This makes it unique for a given knot.

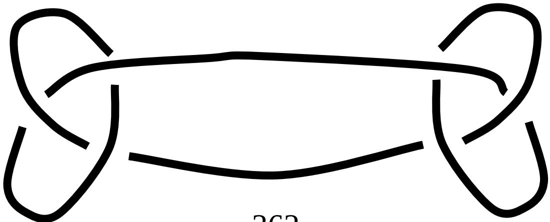
The Alexander Polynomial plays the same role with Alexander Modules as the Alexander Number does to the Alexander Group. It's a single object (polynomial or integer) that's a convenient invariant which can be used in place of the algebraic structure (module or group). However the single object is not quite as powerful an invariant as the algebraic structure.

Putting $\lambda = -1$ the Alexander Module collapses down to the Alexander Group and, if the Alexander Polynomial is $P(\lambda)$ the Alexander number is $P(-1)$. This is because the relation at each crossing is

$$\lambda(\mathbf{a} + \mathbf{b}) + \mathbf{c} + \mathbf{d} = 0$$

for the module and $\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$ for the group.

Example 7:



Alexander Module:
[**x**, **y** | $(\lambda^2 - \lambda + 1)$**x** = **0**, $(\lambda^2 - \lambda + 1)$**y** = **0**]

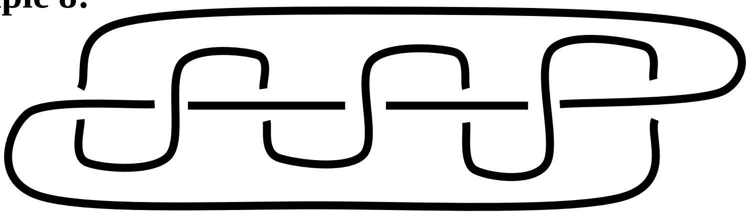
Alexander Matrix: $\begin{bmatrix} \lambda^2 - \lambda + 1 & 0 \\ 0 & \lambda^2 - \lambda + 1 \end{bmatrix}$.

Alexander Polynomial: $(\lambda^2 - \lambda + 1)^2$.

Alexander Group: $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Alexander Number: 9.

Example 8:



Alexander Module:
[**x** | $(\lambda^8 - \lambda^7 + \lambda^6 - \lambda^5 + \lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1)$**x** = **0**].

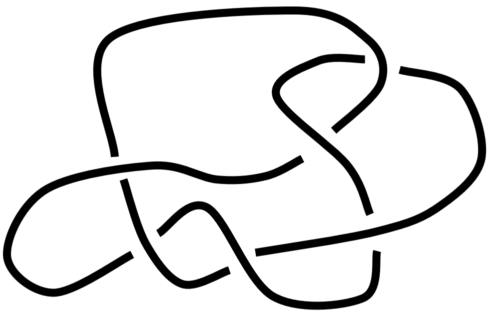
Alexander Matrix:
$[\lambda^8 - \lambda^7 + \lambda^6 - \lambda^5 + \lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1]$, a 1×1 matrix

Alexander Polynomial:
$\lambda^8 - \lambda^7 + \lambda^6 - \lambda^5 + \lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1$.

Alexander Group: $\mathbb{Z}_9$.

Alexander Number: 9.

Example 9:



Alexander Module: [**x** | ($2\lambda^2 - 5\lambda + 2$)**x** = **0**].
Alexander Matrix: [$2\lambda^2 - 5\lambda + 2$].
Alexander Polynomial: $2\lambda^2 - 5\lambda + 2$.
Alexander Group: $\mathbb{Z}_9$.
Alexander Number: 9.

Theorem 3: If P(λ) is the Alexander polynomial of a knot then P(1) = ±1.
Proof: The absolute value of P(1) is the order of the abelian group obtained by putting λ = 1 in the Alexander Module. This is equivalent to using the relation

$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$$

at each crossing.

Because these relations are symmetric in the generators involved the resulting group will not change if under and over passes are interchanged at some of the crossings. But by suitably changing over and under passes at certain crossings the knot can be transformed into the unknot. So this group will be the trivial group. Hence |P(1)| = 1.

A polynomial $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$ of degree n is **symmetric if** $a_n = a_0$, $a_{n-1} = a_1$, …

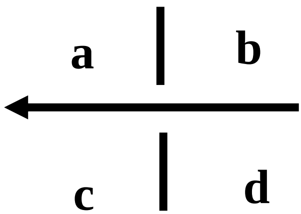
Example 10: The polynomial $7\lambda^4 + 2\lambda^3 - 5\lambda^2 + 2\lambda + 7$ is symmetric.

Note that the powers with the same coefficient correspond. Don't just look at the sequence of non-zero coefficients.

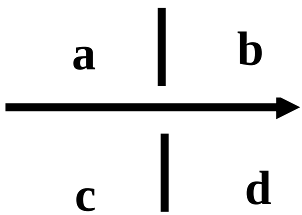
Example 11: $7\lambda^6 + 2\lambda^5 - 5\lambda^4 + 2\lambda^2 + 7$ is not symmetric.

Theorem 4: The Alexander Polynomial of a knot has even degree and is symmetric.
Proof: We omit the proof that the degree is even.
Changing the orientation is equivalent to replacing λ by λ^{-1} in all our calculations and hence in the Alexander Polynomial.

$\boldsymbol{\lambda(a + b) + (c + d) = 0}$



$\boldsymbol{\lambda^{-1}(a + b) + (c + d) = 0}$

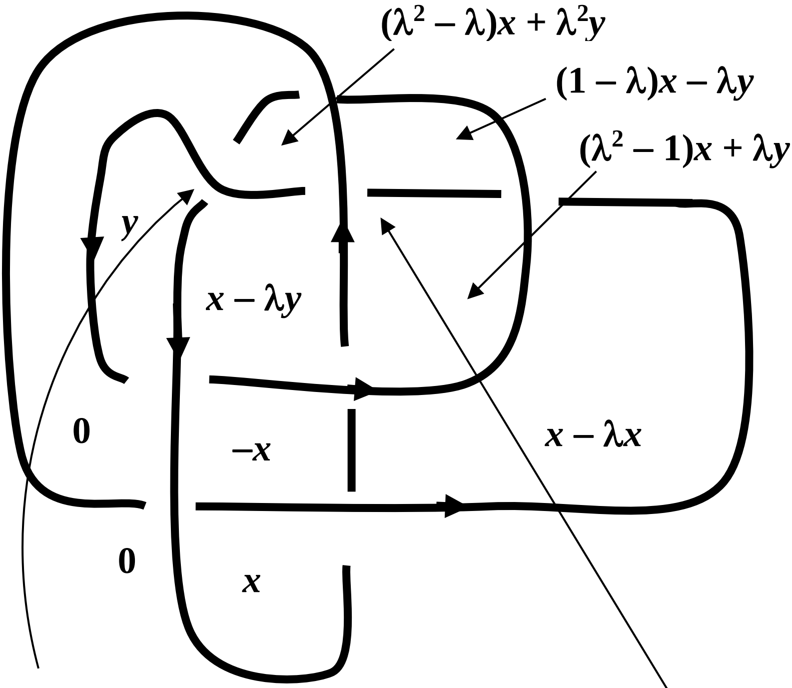


So if $P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$ is the Alexander Polynomial of an oriented knot, the Alexander Polynomial when the orientation is reversed is the Laurent Polynomial $P(\lambda^{-1})$, made into an ordinary polynomial by multiplying by $\pm \lambda^n$.
But the Alexander Polynomial is independent of the orientation (we stated it but omitted the proof). Hence

$$P(\lambda) = (a_n\lambda^{-n} + a_{n-1}\lambda^{-(n-1)} + \ldots + a_1\lambda^{-1} + a_0)\lambda^n$$

$$= \pm(a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n).$$

Example 12:



Here $(\lambda^3 - \lambda^2 + 1)\boldsymbol{x} + (\lambda^3 - \lambda + 1)\boldsymbol{y} = 0$.

Here $(\lambda^3 - \lambda + 1)\boldsymbol{x} + (\lambda^2 - \lambda)\boldsymbol{y} = 0$.

The Alexander Matrix is therefore

$$[M] = \begin{bmatrix} \lambda^3 - \lambda^2 + 1 & \lambda^3 - \lambda + 1 \\ \lambda^3 - \lambda + 1 & \lambda^2 - \lambda \end{bmatrix}$$

The Alexander Polynomial is |det M|

$= |(\lambda^3 - \lambda^2 + 1)(\lambda^2 - \lambda) - (\lambda^3 - \lambda + 1)(\lambda^3 - \lambda + 1)|$

$= \lambda^6 - \lambda^5 - \lambda^3 - \lambda + 1.$

Note that this has even degree and is symmetric.

The Alexander Group (putting $\lambda = -1$) is $\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \cong \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cong [3] \cong \mathbb{Z}_3$.

Based on the Alexander Number alone we might wonder whether this rather complicated knot is really just one of the Trefoil Knots in disguise. But on examining the Alexander Polynomial we can see that it is not one of those.

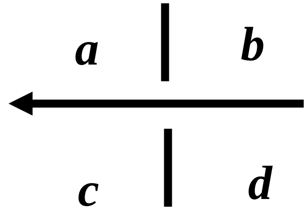
§11.4. Finding the Alexander Polynomial

(1) Choose an orientation for the knot, marking it with arrows.

(2) Assign zero to two adjacent faces of a knot projection and assign a generator to each of the remaining faces.

(3) For each crossing, view it so that the orientation of the overpass is from right to left. If ***a***, ***b***, ***c*** and ***d*** are the generators corresponding to the faces surrounding the crossing, as indicated in the following diagram, introduce the relation:

$$\lambda(\boldsymbol{a} + \boldsymbol{b}) + (\boldsymbol{c} + \boldsymbol{d}) = \boldsymbol{0}.$$



(4) Write down the coefficient matrix for the resulting module.

(5) Take the determinant of this matrix (as a Laurent polynomial in λ).

(6) Multiply by a suitable power of λ so that it becomes an ordinary polynomial with non-zero constant term.

(7) Multiply by -1, if necessary, so that the leading coefficient is positive.

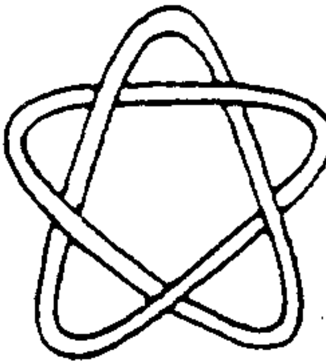
This is the Alexander Polynomial, $P(\lambda)$ of the knot. You can check your answer by seeing that $P(\lambda)$ is symmetric of even degree and $P(1) = \pm 1$.

Before we finish I need to point out that the Alexander polynomial can be derived by other, mostly more complicated methods. Now I believe that the polynomial obtained by the above methods is the same as the classical Alexander Polynomial but I haven't proved this. All I can suggest is that it is a polynomial invariant and it coincides with the classical Alexander Polynomial in all the cases I've considered. It is conceivable, but highly unlikely, that in some cases my Alexander Polynomial may differ from the classical one. However if there is a difference, my version of the Alexander Polynomial would seem to be just as useful as the classical one.

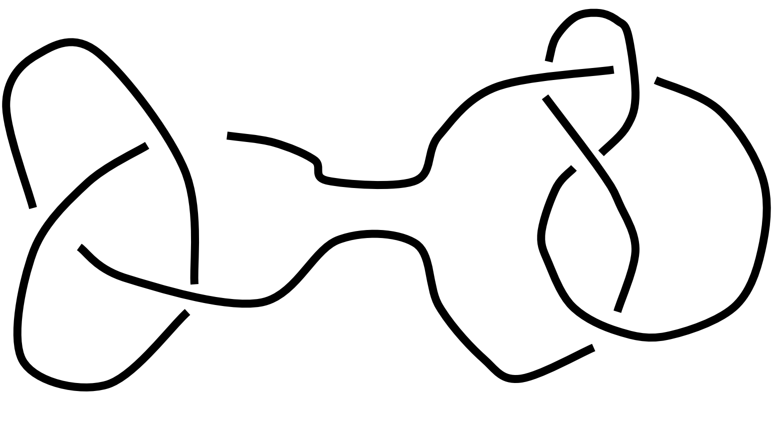
There are other polynomial invariants involving 2 or more variables that are even more discriminating than the Alexander polynomial but they are either harder to calculate or harder to prove that they are invariants.

EXERCISES FOR CHAPTER 11

Exercise 1: Find the Alexander Module and Alexander Polynomial of the following knot:

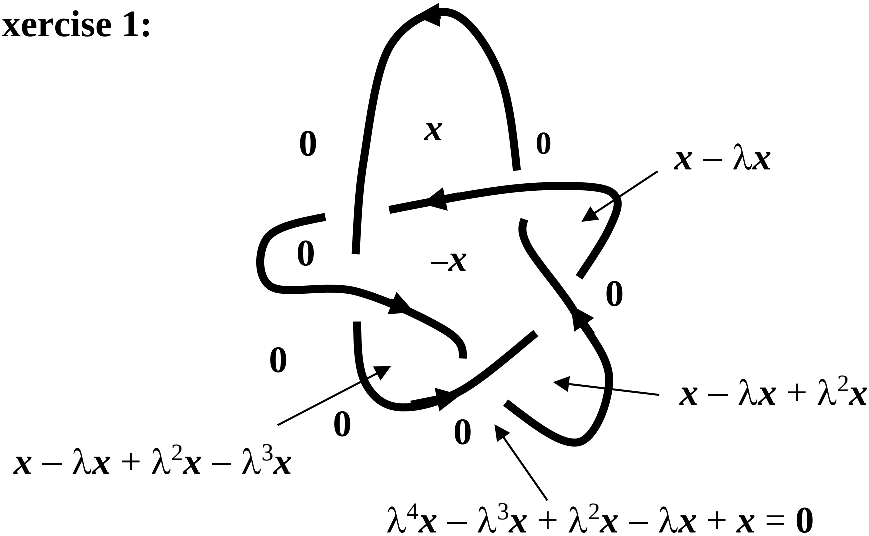


Exercise 2: Find the Alexander Module and Alexander Polynomial of the following knot.



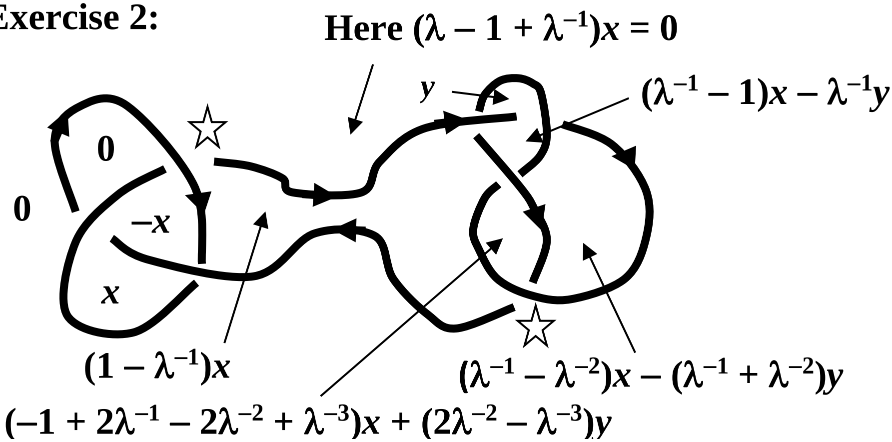
SOLUTIONS FOR CHAPTER 11

Exercise 1:



So the Alexander Polynomial is
$P(\lambda) = \lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1$.
Note that this is Symmetric and $P(1) = 1$.
The Alexander Polynomial is $P(-1) = 5$.

Exercise 2:



At the two crossings marked with a star ☆ we get the two equations:

$$(\lambda^2 - \lambda + 1)\boldsymbol{x} = \mathbf{0} \text{ and}$$

$$(\lambda^3 - 4\lambda^2 + 4\lambda - 1)\boldsymbol{x} + (\lambda^2 - 3\lambda + 1)\boldsymbol{y} = \mathbf{0}.$$

The Alexander Module is

$$\begin{bmatrix} \lambda^2 - \lambda + 1 & 0 \\ \lambda^3 - 4\lambda^2 + 4\lambda - 1 & \lambda^2 - 3\lambda + 1 \end{bmatrix}.$$

The Alexander Polynomial is thus

$$(\lambda^2 - \lambda + 1)(\lambda^2 - 3\lambda + 1) = \lambda^4 - 4\lambda^3 + 5\lambda^2 - 4\lambda + 1.$$